CLOSURE OF THE CONE OF SUMS OF $2d$-POWERS IN REAL TOPOLOGICAL ALGEBRAS

MEHDI GHASEMI[1], SALMA KUHLMANN[2]

ABSTRACT. Let R be a unitary commutative $\mathbb{R}$-algebra and $K \subseteq \mathcal{X}_R := \mathrm{Hom}(R, \mathbb{R})$, closed with respect to the product topology. We consider R endowed with the topology $\mathcal{T}_K$, induced by the family of seminorms $\rho_\alpha(a) := |\alpha(a)|$, for $\alpha \in K$ and $a \in R$. In case K is compact, we also consider the topology induced by $\|a\|_K := \sup_{\alpha\in K} |\alpha(a)|$ for $a \in R$. If K is Zariski dense, then those topologies are Hausdorff. In this paper we prove that the closure of the cone of sums of $2d$-powers, $\sum R^{2d}$, with respect to those two topologies is equal to $\mathrm{Psd}(K) := \{a \in R : \alpha(a) \geq 0, \text{ for all } \alpha \in K\}$. In particular, any continuous linear functional L on the polynomial ring $R = \mathbb{R}[\underline{X}] := \mathbb{R}[X_1, \dots, X_n]$ with $L(h^{2d}) \geq 0$ for each $h \in \mathbb{R}[\underline{X}]$ is integration with respect to a positive Borel measure supported on K. Finally we give necessary and sufficient conditions to ensure the continuity of a linear functional with respect to those two topologies.

1. Introduction

The (real) multidimensional K-moment problem for a given closed set $K \subseteq \mathbb{R}^n$, is the question of when a real valued linear functional L, defined on the real algebra of polynomials $\mathbb{R}[\underline{X}]$, is representable as integration with respect to a positive Borel measure on K. A subset C of $\mathbb{R}[\underline{X}]$ is called a cone, if $C + C \subseteq C$ and $\mathbb{R}^+ C \subseteq C$, where $\mathbb{R}^+$ denotes the non-negative real numbers. Let us denote the cone of non-negative polynomials on K by $\mathrm{Psd}(K)$. If L is representable by a measure then clearly, for any polynomial $f \in \mathrm{Psd}(K)$, $L(f) \geq 0$ (i.e. $L(\mathrm{Psd}(K)) \subseteq \mathbb{R}^+$). Haviland [10, 11], proved that this necessary condition is also sufficient. However, $\mathrm{Psd}(K)$ is seldom finitely generated [19, Proposition 6.1]. So in general, there is no practical

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decision procedure for the membership problem for $\mathrm{Psd}(K)$, and a fortiori for $L(\mathrm{Psd}(K)) \subseteq \mathbb{R}^+$.

We are mainly interested in the solutions of

$$\mathrm{Psd}(K) \subseteq \overline{C}^{\tau}, \tag{1}$$

where C is a cone, K is a closed subset of $\mathbb{R}^n$ and $\overline{C}^{\tau}$ denotes the closure of C with respect to a locally convex topology τ on $\mathbb{R}[\underline{X}]$. It is proved in [8, Proposition 3.1] that (1) holds if and only if for every τ-continuous linear functional L, nonnegative on C, there exists a positive Borel measure μ supported on K, such that

$$L(f) = \int_K f \, d\mu, \quad \forall f \in \mathbb{R}[\underline{X}].$$

Clearly, if the functional L is representable by a measure, then L has to be *positive semidefinite*, i.e., $L(p^2) \geq 0$ for all $p \in \mathbb{R}[\underline{X}]$. So in (1), the simplest cone to consider is $C = \sum \mathbb{R}[\underline{X}]^2$. If τ is the finest locally convex topology φ on $\mathbb{R}[\underline{X}]$, then any given linear functional is φ-continuous [4]. In this setting, Schmüdgen in [20, Theorem 3.1] and Berg, Christensen and Jensen in [2, Theorem 3] prove that $\overline{\sum \mathbb{R}[\underline{X}]^2}^{\varphi} = \sum \mathbb{R}[\underline{X}]^2$. Later, in [3, Theorem 9.1] Berg, Christensen and Ressel prove that taking $C = \sum \mathbb{R}[\underline{X}]^2$ and τ to be the ℓ_1-norm topology, then $K = [-1,1]^n$ solves (1), i.e., $\overline{\sum \mathbb{R}[\underline{X}]^2}^{\|\cdot\|_1} = \mathrm{Psd}([-1,1]^n)$. This was further generalized in [4] and [5] to include commutative semigroup-rings and topologies induced by absolute values. These results has been revisited in [16] with a different approach, and were recently generalized in [8] to weighted ℓ_p-norms, $p \geq 1$. In [9] it is shown that the general result in [5] carries to the even smaller cone of sums of $2d$-powers, $\sum \mathbb{R}[\underline{X}]^{2d} \subseteq \sum \mathbb{R}[\underline{X}]^2$, where $d \geq 1$ is an integer. In particular, $\overline{\sum \mathbb{R}[\underline{X}]^{2d}}^{\|\cdot\|_1} = \mathrm{Psd}([-1,1]^n)$.

We now discuss (1) for other special cones. A set $T \subseteq \mathbb{R}[\underline{X}]$ is called a *preordering*, if $T + T \subseteq T$, $T \cdot T \subseteq T$ and $\sum \mathbb{R}[\underline{X}]^2 \subseteq T$. For $S \subseteq \mathbb{R}[\underline{X}]$, we denote the smallest preordering, containing S by T_S. A preordering T is said to be finitely generated, if $T = T_S$ for some finite set $S \subseteq \mathbb{R}[\underline{X}]$. The smallest preordering of $\mathbb{R}[\underline{X}]$ is $\sum \mathbb{R}[\underline{X}]^2$, as considered above. A subset $M \subseteq \mathbb{R}[\underline{X}]$ is a $\sum \mathbb{R}[\underline{X}]^{2d}$-*module* if $1 \in M$, $M + M \subseteq M$ and $\sum \mathbb{R}[\underline{X}]^{2d} \cdot M \subseteq M$. If $d = 1$, M is said to be a *quadratic* module. M is said to be finitely generated, if $M = M_S$ for some finite set $S \subseteq \mathbb{R}[\underline{X}]$, and *Archimedean* if for every $f \in \mathbb{R}[\underline{X}]$ there exists $n \in \mathbb{N}$ such that $n \pm f \in M$. The non-negativity set of a subset $S \subset \mathbb{R}[\underline{X}]$ will be denoted by K_S, and is defined by $K_S := \{x \in \mathbb{R}^n : \forall f \in S \ f(x) \geq 0\}$. If S is finite, K_S is called a *basic closed semialgebraic set.*

In [21] Schmüdgen proves that for a finite S, if K_S is *compact*, then K_S solves (1) for $C = T_S$ and $\tau = \varphi$, i.e., $\overline{T_S}^{\varphi} = \mathrm{Psd}(K_S)$. Jacobi proved [12] that if M is an Archimedean $\sum \mathbb{R}[\underline{X}]^{2d}$-module then $\overline{M}^{\varphi} = \mathrm{Psd}(K_M)$ (see [18, Theorem 1.3 and 1.4] for $d = 1$). In [15], Lasserre proves that for a specific fixed norm $\|\cdot\|_w$, and any finite S, $\overline{M_S}^{\|\cdot\|_w} = \overline{T_S}^{\|\cdot\|_w} = \mathrm{Psd}(K_S)$. In particular, if $S = \varnothing$, $\overline{\sum \mathbb{R}[\underline{X}]^2}^{\|\cdot\|_w} = \mathrm{Psd}(\mathbb{R}^n)$. It is worth noting that the latter equality could be derived by suitable modifications of Schmüdgen [20, Lemma 6.1 and Lemma 6.4].

Throughout the paper the algebras under consideration are unitary and commutative. In this paper, we study (1) in a more general context. In Section 2, we recall some standard notations and elementary material which will be needed in the following sections. We consider a $\mathbb{Z}[\frac{1}{2}]$-algebra R and a $K \subseteq \mathcal{X}_R := \mathrm{Hom}(R, \mathbb{R})$, closed with respect to the product topology.

In Section 3, we associate to K a topology $\mathcal{T}_K$ on R, making all homomorphisms in K continuous. When K is compact we define a seminorm $\|\cdot\|_K$ on R, which induces a strictly finer topology than $\mathcal{T}_K$. If K is Zariski dense, then those topologies are Hausdorff.

In section 4, we study (1) in terms of the two topologies $\mathcal{T}_K$, $\|\cdot\|_K$ for the cone $C = \sum R^{2d}$ of sums of $2d$-powers. The two main results are Theorems 4.2 and 4.5: we prove that for K as above, the closure of $\sum R^{2d}$ with respect to $\mathcal{T}_K$ is $\mathrm{Psd}(K)$. Here $\mathrm{Psd}(K) := \{a \in R : \alpha(a) \geq 0,\ \text{for all } \alpha \in K\}$. When K is compact, we use Stone-Weierstrass to prove that the closure of $\sum R^{2d}$ with respect to the $\|\cdot\|_K$-topology is again $\mathrm{Psd}(K)$. In case $R = \mathbb{R}[\underline{X}]$, $K = \mathbb{R}^n$ and $d = 1$, the first result is a special case of Schmüdgen's result for locally multiplicatively convex topologies [20, Proposition 6.2], and the second result is straightforward, as noted in [2, Remark 3.2].

Finally, we apply our results to obtain representation of continuous functionals by measures (Corollaries 4.3 and 4.6).

In Section 5, we study the case when R is an $\mathbb{R}$-algebra. We define $\sum R^{2d}$-modules and archimedean modules exactly as we did for $R = \mathbb{R}[\underline{X}]$. We prove that the closure of $\sum R^{2d}$ with respect to any sub-multiplicative norm is $\mathrm{Psd}(\mathcal{K}_{\|\cdot\|})$, where $\mathcal{K}_{\|\cdot\|}$ is the Gelfand spectrum of $(R, \|\cdot\|)$ (see Theorem 5.3). Our proof is algebraic and uses a result of T. Jacobi [12, Theorem 4]. Again, we get representation of continuous functionals by measures (Corollary 5.4).

Next, we study the case where the cone is a $\sum R^{2d}$-module $M \subseteq R$. We define $\mathcal{K}_M = \{\alpha \in \mathcal{X}_R : \alpha(a) \geq 0 \quad \forall a \in M\}$. We show that $\overline{M}^{\mathcal{T}_{\mathcal{X}_R}}$, the closure of M with respect to $\mathcal{T}_{\mathcal{X}_R}$, is $\mathrm{Psd}(\mathcal{K}_M)$ (see Theorem 5.5).

In Section 6, we apply all these results to the ring of polynomials $\mathbb{R}[\underline{X}]$. Moreover, we study the case when K is not necessarily Zariski-dense. We show that if K is contained in a variety, a locally convex and Hausdorff topology τ_K can still be defined, as the limit of an inverse family of topologies on $\mathbb{R}[\underline{X}]$. We show that $\overline{\sum\mathbb{R}[\underline{X}]^{2d}}^{\tau_K} = \tilde{\mathrm{Psd}}(K)$, where $\tilde{\mathrm{Psd}}(K)$ is the set of polynomials which are nonnegative on some open set containing K. Finally, we compare the topologies $\|\cdot\|_K$ and $\mathcal{T}_K$ on $\mathbb{R}[\underline{X}]$ to sub-multiplicative norm topologies, and to the Lasserre's topology $\|\cdot\|_w$, considered on [15].

2. Preliminaries on topological vector spaces and rings.

In the following, all vector spaces are over the field of real numbers (unless otherwise specified). A *topological vector space* is a vector space X equipped with a topology such that the vector space operations (i.e. scalar multiplication and vector summation) are continuous. A subset $A \subseteq X$ is said to be *convex* if for every $x, y \in A$ and $\lambda \in [0,1]$, $\lambda x + (1-\lambda)y \in A$. A *locally convex* (*lc* for short) topology is a topology which admits a neighborhood basis of convex open sets at each point.

Suppose that in addition X is an $\mathbb{R}$-algebra. A subset $U \subseteq X$ is called a *multiplicative set*, an *m-set* for short, if $U\cdot U \subseteq U$. A locally convex topology on X is said to be *locally multiplicatively convex* (or *lmc* for short) if there exists a fundamental system of neighborhoods for 0 consisting of m-sets. It is immediate from the definition that the multiplication is continuous in a lmc-topology.

Definition 2.1. A function $\rho : X \to \mathbb{R}^{\geq 0}$ is called a seminorm, if

(i) $\forall x, y \in X \quad \rho(x+y) \leq \rho(x) + \rho(y)$,

(ii) $\forall x \in X\ \forall r \in \mathbb{R} \quad \rho(rx) = |r|\rho(x)$.

ρ is called a multiplicative seminorm, if in addition ρ satisfies the following

(iii) $\forall x, y \in X \quad \rho(x\cdot y) \leq \rho(x)\rho(y)$.

Definition 2.2. Let $\mathcal{F}$ be a nonempty family of seminorms on X. The topology generated by $\mathcal{F}$ on X is the coarsest topology on X making all seminorms in $\mathcal{F}$ continuous. It is a locally convex topology on X. The family of sets of the form

$$U^{\epsilon}_{\rho_1,\dots,\rho_k}(x) := \{y \in X\ :\ \rho_i(x-y) \leq \epsilon,\ i = 1,\dots,k\}$$

where $\epsilon > 0$ and $\rho_1 \dots, \rho_k \in \mathcal{F}$, forms a basis for this topology.

We have the following characterization of lc and lmc spaces.

Theorem 2.3. *Let X be an algebra and τ a topology on X. Then*

(1) *τ is lc if and only if it is generated by a family of seminorms on X.*
(2) *τ is lmc if and only if it is generated by a family of multiplicative seminorms on X.*

Proof. See [13, Theorem 6.5.1] for (1) and [1, 4.3-2] for (2). □

Let R be a commutative ring with 1 and $\frac{1}{2} \in R$. We always assume that $\mathcal{X}_R = \mathrm{Hom}(R, \mathbb{R})$, the set of unitary homomorphisms, is nontrivial. Clearly, $\mathcal{X}_R \subset \mathbb{R}^R$, therefore, it carries a topology as subspace of $\mathbb{R}^R$ with the product topology which is Hausdorff. For any $a \in R$ let $U(a) := \{\alpha \in \mathcal{X}_R : \alpha(a) < 0\}$. The family $\{U(a) : a \in R\}$ forms a subbasis for the subspace topology on $\mathcal{X}_R$ which is the coarsest topology making all projection functions $\hat{a} : \mathcal{X}_R \to \mathbb{R}$ continuous where for $a \in R$, $\hat{a}$ is defined by $\hat{a}(\alpha) = \alpha(a)$. $\mathcal{X}_R$ also can be embedded in $\mathrm{Sper}(R)$ equipped with spectral topology [17, Theorem 5.2.5 and Lemma 5.2.6]. Since all projections are continuous, the topology of $\mathcal{X}_R$ coincides with the subspace topology inherited from $\mathbb{R}^R$ equipped with product topology. For K be a subset of $\mathcal{X}_R$ we denote by $C(K)$ the algebra of continuous real valued functions on K.

Definition 2.4. To any subset S of R we associate a subset $\mathcal{Z}(S)$ of $\mathcal{X}_R$, called the zeros of S or the variety of S by $\mathcal{Z}(S) := \{\alpha \in \mathcal{X}_R : \alpha(S) = \{0\}\}$. Denote by $\langle S \rangle$ the ideal generated by S. Then $\mathcal{Z}(S) = \mathcal{Z}(\langle S \rangle)$, and the family $\{\mathcal{X}_R \setminus \mathcal{Z}(S) : S \subseteq R\}$ forms a sub-basis for a topology called the *Zariski topology* on $\mathcal{X}_R$.

Now Suppose that the map (defined in Lemma 3.1 with $K = \mathcal{X}_R$, $a \mapsto \hat{a}$ is injective, then a subset K of $\mathcal{X}_R$ is dense in $\mathcal{X}_R$ with respect to the Zariski topology (*Zariski dense*) if and only if $K \nsubseteq \mathcal{Z}(I)$ for any proper I ideal of R.

Example 2.5. Suppose that $K \subseteq \mathbb{R}^n$ has nonempty interior, then clearly K is Zariski dense in $\mathbb{R}^n$. Let K be a basic closed semialgebraic set, i.e., $K := \{x \in \mathbb{R}^n : f_i(x) \geq 0, i = 1, \ldots, m\}$ where $f_i \in \mathbb{R}[\underline{X}]$, $i = 1, \ldots, m$. We show that if K has an empty interior then K is contained in $\mathcal{Z}(g)$, where $g = f_1 \ldots f_m$. For each $x \in K$, $f_i(x) = 0$ for some $1 \leq i \leq m$. Otherwise, $f_i(x) > 0$ for each $i = 1, \ldots, m$, the continuity of polynomials implies that $f_i(y) > 0$, $i = 1, \ldots, m$ for y sufficiently close to x and hence x is an interior point which is impossible. Therefore $g(x) = 0$ and hence $K \subseteq \mathcal{Z}(g)$. Note that any closed semialgebraic set K, is a finite union of basic closed semialgebraic sets [6,

Theorem 2.7.2] i.e., $K = \bigcup_{i=1}^{l} K_i$. If K has an empty interior, then each K_i is so and hence $K \subseteq \mathcal{Z}(g_1 \dots g_l)$, where each g_i is the product of generators of K_i.

Remark 2.6. In general, the conclusion of the above example is false. For example, let $R = \mathbb{Z}_3 \times \mathbb{R}[\underline{X}]$ with component wise addition and multiplication. R is a $\mathbb{Z}[\frac{1}{2}]$-module where $\frac{1}{2}(1,r) = (2,\frac{r}{2})$ and $\frac{1}{2}(2,r) = (1,\frac{r}{2})$. Also $\mathcal{X}_R = \mathbb{R}^n$ is non-trivial. $I = \mathbb{Z}_3 \times \{0\}$ is a proper ideal of R and $\mathcal{Z}(I) = \mathbb{R}^n$. Hence, every semialgebraic set is contained in $\mathcal{Z}(I)$.

Definition 2.7. If R is a ring, a function $N : R \to \mathbb{R}^+$ is called a *ring-seminorm* if the following conditions hold for all $x, y \in R$:

(i) $N(0) = 0$,
(ii) $N(x+y) \leq N(x) + N(y)$,
(iii) $N(-x) = N(x)$,
(iv) $N(xy) \leq N(x)N(y)$.

N is called a *ring-norm*, if in addition

(v) $N(x) = 0$ only if $x = 0$.

We close this section by stating a general version of Haviland's Theorem.

Theorem 2.8. *Suppose R is an $\mathbb{R}$-algebra, X is a Hausdorff space, and $\hat{}: R \to C(X)$ is an $\mathbb{R}$-algebra homomorphism such that for some $p \in R$, $\hat{p} \geq 0$ on X and the set $X_i = \hat{p}^{-1}([0,i])$ is compact for each $i = 1, 2, \dots$. Then for every linear functional $L : R \to \mathbb{R}$ satisfying*

$$L(\{a \in R : \hat{a} \geq 0 \text{ on } X\}) \subseteq \mathbb{R}^+,$$

there exists a Borel measure μ on X such that $\forall a \in R \quad L(a) = \int_X \hat{a}\, d\mu$.

Proof. See [17, Theorem 3.2.2]. □

3. The topologies $\mathcal{T}_K$ and $\|.\|_K$.

Throughout we assume that the map $\hat{} : R \to C(\mathcal{X}_R)$, defined by $\hat{a}(\alpha) = \alpha(a)$ is injective.

Lemma 3.1. *Let K be a subset of $\mathcal{X}_R$, then*

1. *The map $\Phi : R \to C(K)$ defined by $\Phi(a) = \hat{a}|_K$ is a homomorphism.*
2. *$Im(\Phi)$ contains a copy of $\mathbb{Z}[\frac{1}{2}]$.*

Proof. (1) This is clear. Let $a, b \in R$, then for each $\alpha \in K$ we have

$$\begin{array}{rcl} \Phi(a+b)(\alpha) & = & \widehat{(a+b)}|_K(\alpha) \\ & = & \alpha(a+b) \\ & = & \alpha(a)+\alpha(b) \\ & = & \hat{a}|_K(\alpha)+\hat{b}|_K(\alpha) \\ & = & \Phi(a)(\alpha)+\Phi(b)(\alpha). \end{array}$$

Similarly $\Phi(a \cdot b) = \Phi(a) \cdot \Phi(b)$.
(2) Since $\mathcal{X}_R$ consists of unitary homomorphisms, $\hat{1}(\alpha) = \alpha(1) = 1$, so the constant function $1 \in \Phi(R)$. Moreover for each $m \in \mathbb{Z}$ and $n \in \mathbb{N}$, $\frac{m}{2^n} \in R$ and $\frac{\hat{m}}{2^n}$ is the constant function $\frac{m}{2^n}$ which belongs to $Im(\Phi)$, so $\mathbb{Z}[\frac{1}{2}] \subseteq Im(\Phi)$. □

The topology $\mathcal{T}_K$. Let $K \subseteq \mathcal{X}_R$. To any $\alpha \in K$ we associate a seminorm ρ_α on $C(K)$ by defining $\rho_\alpha(f) := |f(\alpha)|$ for $f \in C(K)$. Note that $\rho_\alpha(fg) = \rho_\alpha(f)\rho_\alpha(g)$, so ρ_α is a multiplicative seminorm. The family of seminorms $\mathcal{F}_K = \{\rho_\alpha : \alpha \in K\}$ thus induces an lmc-topology on $C(K)$.

Similarly, The restriction of ρ_α to $\Phi(R)$ induces a multiplicative ring-seminorm on R by defining $\rho_\alpha(a) := |\hat{a}(\alpha)| = |\alpha(a)|$ for $a \in R$. Thus family of ring-seminorms $\mathcal{F}_K$ induces a topology $\mathcal{T}_K$ on R.

To ease the notation we shall denote the neighborhoods by

$$U^\epsilon_{\rho_{\alpha_1},\dots,\rho_{\alpha_m}}(a) := U^\epsilon_{\alpha_1,\dots,\alpha_m}(a)\ .$$

Remark 3.2. $\mathcal{T}_K$ is the coarsest topology on R for which all $\alpha \in K$ are continuous. $\mathcal{T}_K$ is also the coarsest topology on R, for which Φ is continuous. This is clear, because $\rho_\alpha(a) = \rho_\alpha(\Phi(a))$ for each $a \in R$.

We note for future reference that the topology generated by $\mathcal{F}_K$ on $C(K)$ is Hausdorff [4, Proposition 1.8].

Theorem 3.3. *Let $K \subseteq \mathcal{X}_R$ and $\Phi : R \to C(K)$ be the map defined in Lemma 3.1. The following are equivalent:*

1. *K is Zariski dense,*
2. $\ker \Phi = \{0\}$,
3. *$\mathcal{T}_K$ is a Hausdorff topology.*

Proof. (1)⇒(2) In contrary, suppose that $\ker \Phi \neq \{0\}$ and let $0 \neq a \in \ker \Phi$. Then by definition, $\hat{a}(\alpha) = 0$ for all $\alpha \in K$. This implies $K \subseteq \mathcal{Z}(a)$ which contradicts the assumption that K is Zariski dense.
(2)⇒(3) Since Φ is injective, by Remark 3.2, Φ is a topological embedding. This implies that $\mathcal{T}_K$ is Hausdorff as well.

(3)⇒(1) Suppose that K is not Zariski dense. Than $K \subseteq \mathcal{Z}(I)$ for a nontrivial ideal of R. Take $a, b \in I$, $a \neq b$ and let U be an open set in $\mathcal{T}_K$, containing a. By definition, there exist $\alpha_1, \ldots, \alpha_m \in K$ and $\epsilon > 0$ such that

$$U^{\epsilon}_{\alpha_1,\ldots,\alpha_m}(a) \subseteq U.$$

For each $i = 1, \ldots, m$, $\rho_{\alpha_i}(a-b) = |\alpha_i(b-a)| = 0$, therefore $b \in U^{\epsilon}_{\alpha_1,\ldots,\alpha_m}(a)$ and hence $b \in U$. This shows that $\mathcal{T}_K$ is not Hausdorff, a contradiction. □

The topology $\|.\|_K$. Assume now that K is compact. In this case, $C(K)$ carries a natural norm topology, the norm defined by $\|f\|_K = \sup_{\alpha \in K} |f(\alpha)|$. The inequalities $\|f+g\|_K \leq \|f\|_K + \|g\|_K$ and $\|fg\|_K \leq \|f\|_K \|g\|_K$ implies the continuity of addition and multiplication on $(C(K), \|\cdot\|_K)$.

Lemma 3.4. *If $K \subseteq \mathcal{X}_R$ is compact, then $\Phi(R)$ is dense in $(C(K), \|\cdot\|_K)$.*

Proof. Let $\mathcal{A} = \overline{\Phi(R)}$. We make use of Stone-Weierstrass Theorem to show that $\mathcal{A} = C(K)$. K is compact and Hausdorff, so once we show that $\mathcal{A}$ is an $\mathbb{R}$-algebra which contains all constant functions and separates points of K, we are done (See [22, Theorem 44.7]). Note that $\mathcal{A}$ contains all constant functions because $\mathbb{Z}[\frac{1}{2}] \subset \Phi(R)$ which is dense in $\mathbb{R}$. Since addition and multiplication are continuous, they extend continuously to $\mathcal{A}$, therefore, $\mathcal{A}$ is also closed under addition and multiplication. Moreover, $\mathcal{A}$ separates points of K, because $\Phi(R)$ does. Hence, by Stone-Weierstrass Theorem $\mathcal{A} = C(K)$. □

Remark 3.5.

1. Corresponding to Remark 3.2, defining the ring-norm $\|\cdot\|_K$ on R by $\|a\|_K = \|\hat{a}\|_K$ induces a topology which is the coarsest topology such that Φ is continuous. But $\|\cdot\|_K$ is not necessarily a norm, unless when Φ is injective which by Theorem 3.3 is equivalent to K being Zariski dense.
2. For any $\alpha \in K$, the evaluation map at α, over $C(K, \|\cdot\|_K)$ satisfies the inequality $|f(\alpha)| = \rho_\alpha(f) \leq \|f\|_K$, so it is continuous for each $\alpha \in K$. This observation shows that each $\mathcal{T}_K$-open set is also $\|\cdot\|_K$-open, i.e., $\|\cdot\|_K$-topology is finer than $\mathcal{T}_K$. We show that if K is infinite, then $\|\cdot\|_K$-topology is strictly finer than $\mathcal{T}_K$.

Proposition 3.6. *If K is an infinite, compact subset of $\mathcal{X}_R$, then $\|\cdot\|_K$-topology is strictly finer than $\mathcal{T}_K$.*

Proof. Let $\alpha_1, \dots, \alpha_m \in K$ and $0 < \epsilon < 1$. We claim that there exists $a \in R$ such that $a \in U^{\epsilon}_{\alpha_1,\dots,\alpha_m}(0)$ and $\|a\|_K > \epsilon$. Note that $\mathcal{X}_R$ is Hausdorff and so is K. Compactness of K implies that K is a normal space. Take $A = \{\alpha_1, \dots, \alpha_m\}$ and $B = \{\beta\}$, where $\beta \in K \setminus A$. By Urysohn's lemma, there exists a continuous function $f : K \to [0,1]$ such that $f(A) = 0$ and $f(\beta) = 1$. For $\delta < \min\{\epsilon, 1-\epsilon\}$, there exists $a \in R$ such that $\|f - a\|_K < \delta$ by Lemma 3.4. Clearly $a \in U^{\epsilon}_{\alpha_1,\dots,\alpha_m}(0)$ and $|\hat{a}(\beta)| > \epsilon$ which implies that $\|a\|_K > \epsilon$ which completes the proof of the claim.

Let $N_\epsilon(0) = \{a \in R : \|a\|_K < \epsilon\}$ be an open ball around 0 in $\|\cdot\|_K$ for $0 < \epsilon < 1$. We show that $N_\epsilon(0)$ does not contain any open neighborhood of 0 in $\mathcal{T}_K$. In contrary, suppose that $0 \in U^{\delta}_{\alpha_1,\dots,\alpha_m}(0) \subseteq N_\epsilon(0)$. Obviously $\delta \leq \epsilon$ and so there exists $a \in U^{\delta}_{\alpha_1,\dots,\alpha_m}(0)$ such that $a \notin N_\epsilon(0)$ which is a contradiction. So, $N_\epsilon(0)$ is not open in $\mathcal{T}_K$ and hence, $\|\cdot\|_K$-topology is strictly finer that $\mathcal{T}_K$. $\square$

4. Closures of $\sum R^{2d}$ in $\mathcal{T}_K$ and $\|\cdot\|_K$

In this section, we compute the closure $\sum R^{2d}$ in the two topologies defined in the previous section. In particular, for compact $K \subseteq \mathcal{X}_R$, we show that

$$\overline{\sum R^{2d}}^{\|\cdot\|_K} = \overline{\sum R^{2d}}^{\mathcal{T}_K},$$

although for infinite K, the $\|\cdot\|_K$-topology is strictly finer than $\mathcal{T}_K$ on R by Proposition 3.6. Let

$$C^+(K) := \{f : K \to \mathbb{R} \ : \ f \text{ is continuous and } \forall \alpha \in K \quad f(\alpha) \geq 0\},$$

denote the set of nonnegative real valued continuous functions over K and

$$\mathrm{Psd}(K) := \{a \in R : \hat{a} \in C^+(K)\}.$$

Proposition 4.1. *Psd(K) is closed in $\mathcal{T}_K$. If K is compact, then Psd(K) is also closed in $\|\cdot\|_K$-topology.*

Proof. For each $\alpha \in K$, let $e_\alpha(f) = f(\alpha)$ be the evaluation map. Then $e_\alpha^{-1}([0,\infty))$ is closed, by continuity of e_α in $\mathcal{T}_K$. Therefore $\mathrm{Psd}(K) = \bigcap_{\alpha \in K} e_\alpha^{-1}([0,\infty))$ is closed. If K is compact, then, again e_α is $\|\cdot\|_K$-continuous. Therefore $\mathrm{Psd}(K)$ is also closed with respect to $\|\cdot\|_K$. $\square$

Theorem 4.2. *For any compact set $K \subseteq \mathcal{X}_R$ and integer $d \geq 1$, $\overline{\sum R^{2d}}^{\|\cdot\|_K} =$ Psd(K).*

Proof. Since $\sum R^{2d} \subseteq \text{Psd}(K)$ and $\text{Psd}(K)$ is closed, clearly $\overline{\sum R^{2d}}^{\|\cdot\|_K} \subseteq \text{Psd}(K)$. To show the reverse inclusion, let $a \in \text{Psd}(K)$ and $\epsilon > 0$ be given. Since $\hat{a} \geq 0$ on K, $\sqrt[2d]{\hat{a}} \in C(K)$. Continuity of multiplication implies the continuity of the map $f \mapsto f^{2d}$. Therefore, there exists $\delta > 0$ such that $\| \sqrt[2d]{\hat{a}} - f\|_K < \delta$ implies $\|\hat{a} - f^{2d}\|_K < \epsilon$. Using Lemma 3.4, there is $b \in R$ such that $\| \sqrt[2d]{\hat{a}} - \hat{b}\|_K < \delta$ and so $\|\hat{a} - \hat{b}^{2d}\|_K < \epsilon$. By definition, $\hat{a} - \hat{b}^{2d} = \Phi(a - b^{2d})$ and hence $\|a - b^{2d}\|_K < \epsilon$. Therefore, any neighborhood of a has nonempty intersection with $\sum R^{2d}$ which proves the reverse inclusion $\text{Psd}(K) \subseteq \overline{\sum R^{2d}}^{\|\cdot\|_K}$. □

Corollary 4.3. *Let K be a compact subset of $\mathcal{X}_R$, $d \geq 1$ an integer. Assume that $L : R \to \mathbb{R}$ is $\|\cdot\|_K$-continuous, $\mathbb{Z}[\frac{1}{2}]$-linear map, such that $L(a^{2d}) \geq 0$ for all $a \in R$, then there exists a Borel Measure μ on K such that $\forall a \in R$ $L(a) = \int_K \hat{a}\ d\mu$.*

Proof. Let $\hat{R} := \{\hat{a} : a \in R\}$ and define $\bar{L} : \hat{R} \to \mathbb{R}$ by $\bar{L}(\hat{a}) = L(a)$.

We prove if $\hat{a} \geq 0$, then $L(a) \geq 0$. To see this, let $\epsilon > 0$ be given and find $\delta > 0$ such that $\|a - b\|_K < \delta$ implies $|L(a) - L(b)| < \epsilon$. Take $c_\epsilon \in R$ such that $\|a - c_\epsilon^{2d}\|_K < \delta$. Then

$$L(c_\epsilon^{2d}) - \epsilon < L(a) < L(c_\epsilon) + \epsilon,$$

let $\epsilon \to 0$, yields $L(a) \geq 0$.

Note that $\bar{L}$ is well-defined, since $\hat{a} = 0$, implies $\hat{a} \geq 0$ and $-\hat{a} \geq 0$, so $\bar{L}(\hat{a}) \geq 0$ and $\bar{L}(-\hat{a}) \geq 0$, simultaneously and hence $\bar{L}(\hat{a}) = 0$. $\|\cdot\|_K$-continuity of L on R, implies $\|\cdot\|_K$-continuity of $\bar{L}$ on $\hat{R}$. Let A be the $\mathbb{R}$-subalgebra of $C(K)$, generated by $\hat{R}$. Elements of A are of the form $r_1\hat{a_1} + \cdots + r_k\hat{a_k}$, where $r_i \in \mathbb{R}$ and $a_i \in R$, for $i = 1\ldots,k$ and $k \geq 1$. $\bar{L}$ is continuously extensible to A by $\bar{L}(r\hat{a}) := r\bar{L}(\hat{a})$. By Lemma 3.4, $\hat{R}$ and hence A is dense in $(C(X), \|\cdot\|_K)$. Hahn-Banach Theorem gives a continuous extension of $\bar{L}$ to $C(X)$. Denoting the extension again by $\bar{L}$, an easy verification shows that $\bar{L}(C^+(K)) \subseteq \mathbb{R}^+$. Applying Riesz Representation Theorem, the result follows. □

Remark 4.4. For the special case $K = \mathbb{R}^n$ and $R = \mathbb{R}[\underline{X}]$, it follows from [20, Proposition 6.2] that the closure of $\sum \mathbb{R}[\underline{X}]^2$ with respect to the finest lmc topology η_0 on $\mathbb{R}[\underline{X}]$ is equal to $\text{Psd}(\mathbb{R}^n)$. Since $\mathcal{T}_{\mathbb{R}^n}$ is lmc and $\text{Psd}(\mathbb{R}^n)$ is closed in $\mathcal{T}_{\mathbb{R}^n}$ we get

$$\text{Psd}(\mathbb{R}^n) = \overline{\sum \mathbb{R}[\underline{X}]^2}^{\eta_0} \subseteq \overline{\sum \mathbb{R}[\underline{X}]^2}^{\mathcal{T}_{\mathbb{R}^n}} \subseteq \overline{\text{Psd}(\mathbb{R}^n)}^{\mathcal{T}_{\mathbb{R}^n}} = \text{Psd}(\mathbb{R}^n).$$

In the next theorem, we show that a similar result holds for arbitrary K and the smaller set of sums of $2d$-powers $\sum R^{2d} \subset \sum R^2$.

Theorem 4.5. *Let $K \subseteq \mathcal{X}_R$ be a closed set and $d \geq 1$, then $\overline{\sum R^{2d}}^{\mathcal{T}_K} = Psd(K)$.*

Proof. Since $\sum R^{2d} \subseteq \mathrm{Psd}(K)$ and by Proposition 4.1, $\mathrm{Psd}(K)$ is closed, we have $\overline{\sum R^{2d}}^{\mathcal{T}_K} \subseteq \mathrm{Psd}(K)$.

To get the reverse inclusion, let $a \in \mathrm{Psd}(K)$ be given. We show that any neighborhood of a in $\mathcal{T}_K$ has a nonempty intersection with $\sum R^{2d}$.

Claim. If $\hat{a} > 0$ on K then $a \in \overline{\sum R^{2d}}^{\mathcal{T}_K}$.

To prove this, let U be an open set, containing a. There exist $\alpha_1, \ldots, \alpha_n \in K$ and $\epsilon > 0$ such that $a \in U^{\epsilon}_{\alpha_1,\ldots,\alpha_n}(a) \subseteq U$. Chose $m \in \mathbb{N}$ such that $\max_{1\leq i\leq n} \alpha_i(a) < 2^{2dm}$. Now for $b = \frac{a}{2^{2dm}}$ we have $0 < \alpha_i(b) < 1$. By continuity of $f(t) = t^{2d}$, for each $i = 1, \ldots, n$ there exists $\delta_i > 0$ such that for any t, if $|t - \alpha_i(b)^{1/2d}| < \delta_i$, then $|t^{2d} - \alpha_i(b)| < \frac{\epsilon}{2^{2dm}}$. Take $\delta = \min_{1\leq i\leq n} \delta_i$. Let $p(t) = \sum_{j=0}^{N} \lambda_j t^j$ be the real polynomial satisfying $p(\alpha_i(b)) = \sqrt[2d]{\alpha_i(b)}$ for $i = 1, \ldots, n$. Since $\mathbb{Z}[\frac{1}{2}]$ is dense in $\mathbb{R}$ one can choose $\tilde{\lambda}_j \in \mathbb{Z}[\frac{1}{2}]$, such that $|\sum_{j=1}^{N} \lambda_j \alpha_i(b)^j - \sum_{j=1}^{N} \tilde{\lambda}_j \alpha_i(b)^j| < \delta$, for $i = 1, \ldots, n$. Let $c = \sum_{j=1}^{N} \tilde{\lambda}_j b^j \in R$. Then $|\alpha_i(b) - \alpha_i(c)^{2d}| < \frac{\epsilon}{2^{2dm}}$. Multiplying by 2^{2dm}, $|\alpha_i(a) - \alpha_i(2^m c)^{2d}| < \epsilon$ i.e. $\rho_{\alpha_i}(a - (2^m c)^{2d}) < \epsilon$ for $i = 1, \ldots, n$. Therefore $U^{\epsilon}_{\alpha_1,\ldots,\alpha_n}(a) \cap \sum R^{2d} \neq \varnothing$ and hence $a \in \overline{\sum R^{2d}}^{\mathcal{T}_K}$ which completes the proof of the claim.

For an arbitrary $a \in \mathrm{Psd}(K)$, and each $k \in \mathbb{N}$, $\widehat{(a + \frac{1}{2^k})} > 0$ on K, so $\forall k \in \mathbb{N}$, $a + \frac{1}{2^k} \in \overline{\sum R^{2d}}^{\mathcal{T}_K}$. Letting $k \to \infty$, $\rho_\alpha(a + \frac{1}{2^k}) \to \rho_\alpha(a)$, we get $a \in \overline{\sum R^{2d}}^{\mathcal{T}_K}$ and hence $\mathrm{Psd}(K) \subseteq \overline{\sum R^{2d}}^{\mathcal{T}_K}$ as desired. □

Corollary 4.6. *Let K be a closed subset of $\mathcal{X}_R$ and $d \geq 1$ an integer. Assume that there exists $p \in R$, such that $\hat{p} \geq 0$ on K, $K_i = \hat{p}^{-1}([0, i])$ is compact for each i and $L : R \to \mathbb{R}$ is $\mathcal{T}_K$-continuous, $\mathbb{Z}[\frac{1}{2}]$-linear map, and $L(a^{2d}) \geq 0$ for all $a \in R$, then there exists a Borel Measure μ on K such that $\forall a \in R$ $L(a) = \int_K \hat{a}\ d\mu$.*

Proof. Following the argument in the proof of Corollary 4.3, the map $\bar{L} : \hat{R} \to \mathbb{R}$ is well-defined and has a $\mathcal{F}_K$-continuous extension to the $\mathbb{R}$-subalgebra A of $C(K)$, generated by $\hat{R}$. Applying Theorem 2.8 to $\bar{L}$, $\hat{p}$ and A, the result follows. □

5. Results for $\mathbb{R}$-Algebras

In this section we assume that R is an $\mathbb{R}$-algebra. First we consider closure of $\sum R^{2d}$ with respect to any sub-multiplicative norm $\|\cdot\|$ on R. We prove that the closure of $\sum R^{2d}$ with respect to the norm is equal to nonnegative elements over the global spectrum $\mathcal{K}_{\|\cdot\|}$ of $(R, \|\cdot\|)$. Recall that the global spectrum of a topological $\mathbb{R}$-algebra, also known as the Gelfand spectrum, is the set of all continuous elements of $\mathcal{X}_R$.

Furthermore, in the case of $\mathbb{R}$-algebras, we generalize the conclusion of Theorem 4.5 to an arbitrary $\sum R^{2d}$-module M.

Normed $\mathbb{R}$-Algebras. Suppose that $(R, \|\cdot\|)$ is a normed $\mathbb{R}$-algebra, i.e., the norm satisfies the sub-multiplicativity condition $\|xy\| \leq \|x\|\|y\|$ for all $x, y \in R$.

Lemma 5.1. *If $\alpha \in \mathcal{K}_{\|\cdot\|}$ then $|\alpha(x)| \leq \|x\|$, for all $x \in R$.*

Proof. In contrary suppose that $\exists x \in R$ such that $|\alpha(x)| > \|x\|$. Then for $n \geq 1$,

$$\|x^n\| \leq \|x\|^n \leq |\alpha(x)|^n = |\alpha(x^n)|.$$

Therefore $\frac{|\alpha(x)|^n}{\|x\|^n} \leq \frac{|\alpha(x^n)|}{\|x^n\|}$. Since $\frac{|\alpha(x)|}{\|x\|} > 1$, $\frac{|\alpha(x^n)|}{\|x^n\|} \to \infty$ as $n \to \infty$. This contradicts the fact that α is $\|\cdot\|$-continuous. So

$$\forall x \in R \quad |\alpha(x)| \leq \|x\|.$$

□

Lemma 5.2. *Let $d \geq 1$ be an integer, $a \in R$ and $r > \|a\|$. Then $(r \pm a)^{\frac{1}{2d}} \in \tilde{R}$, where $\tilde{R}$ is the completion of $(R, \|\cdot\|)$.*

Proof. Let $\sum_{i=0}^{\infty} \lambda_i t^i$ be the power series expansion on $(r \pm t)^{\frac{1}{2d}}$ about $t = 0$. The series has the radius of convergence r. Therefore, it converges for every t with $|t| < r$. Since $r > \|a\|$, $\sum_{i=0}^{\infty} \lambda_i \|a\|^i < \infty$. Note that $\|a^i\| \leq \|a\|^i$ for each $i = 0, 1, 2, \cdots$, assuming $a^0 = 1$, so

$$\begin{array}{rcl} \|\sum_{i=0}^{\infty} \lambda_i a^i\| & \leq & \sum_{i=0}^{\infty} \lambda_i \|a^i\| \\ & \leq & \sum_{i=0}^{\infty} \lambda_i \|a\|^i \\ & < & \infty. \end{array}$$

This implies that $(1 \pm a)^{\frac{1}{2d}} = \sum_{i=0}^{\infty} \lambda_i a^i \in \tilde{R}$. □

Let $A_{\|\cdot\|} := \{\|x\| \pm x : x \in R\}$ and M_{2d} be the $\sum R^{2d}$-module generated by $A_{\|\cdot\|}$. Clearly, M_{2d} is archimedean and hence $\mathcal{K}_{M_{2d}}$ is compact. Note that

$$\begin{array}{rcl} \alpha \in \mathcal{K}_{M_{2d}} & \Leftrightarrow & \alpha(a) \geq 0 \quad \forall a \in M_{2d} \\ & \Leftrightarrow & \|x\| \pm \alpha(x) \geq 0 \quad \forall x \in R \\ & \Leftrightarrow & |\alpha(x)| \leq \|x\| \quad \forall x \in R \\ & \Leftrightarrow & \alpha \text{ is } \|\cdot\|\text{-continuous.} \end{array}$$

Therefore $\mathcal{K}_{M_{2d}}$ is nothing but global spectrum of $(R, \|\cdot\|)$.

Theorem 5.3 is the analogue of [9, Theorem 4.3] for normed algebras. Note that the fact that the Gelfand spectrum $\mathcal{K}_{\|\cdot\|}$ is compact is well-known (under additional assumptions)(see [14, Theorem 2.2.3]). However our proof is algebraic and based on the following result of T. Jacobi.

Theorem. Suppose $M \subseteq R$ is an archimedean $\sum R^{2d}$-module of R for some integer $d \geq 1$. Then, $\forall\ a \in R$,

$$\hat{a} > 0 \text{ on } \mathcal{K}_M \;\Rightarrow\; a \in M.$$

Proof. See [12, Theorem 4] for every $d \geq 1$ or [18, Theorem 1.4] for $d = 1$. $\square$

Theorem 5.3. *Let $(R, \|\cdot\|)$ be a normed $\mathbb{R}$-algebra and $d \geq 1$ an integer. Then $\mathcal{K}_{\|\cdot\|}$ is compact and $\overline{\sum R^{2d}}^{\|\cdot\|} = Psd(\mathcal{K}_{\|\cdot\|})$.*

Proof. Since each $\alpha \in \mathcal{K}_{\|\cdot\|}$ is continuous and $\mathrm{Psd}(\mathcal{K}_{\|\cdot\|}) = \bigcap_{\alpha \in \mathcal{K}_{\|\cdot\|}} \alpha^{-1}([0,\infty))$, $\mathrm{Psd}(\mathcal{K}_{\|\cdot\|})$ is $\|\cdot\|$-closed. Clearly $\sum R^{2d} \subseteq \mathrm{Psd}(\mathcal{K}_{\|\cdot\|})$, therefore $\overline{\sum R^{2d}}^{\|\cdot\|} \subseteq \mathrm{Psd}(\mathcal{K}_{\|\cdot\|})$.

For the reverse inclusion we have to show that if $a \in \mathrm{Psd}(\mathcal{K}_{\|\cdot\|})$ and $\epsilon > 0$ are given, then $\exists b \in \sum R^{2d}$ with $\|a - b\| \leq \epsilon$. Note that $\hat{a} + \frac{\epsilon}{2}$ is strictly positive on $\mathcal{K}_{\|\cdot\|}$. Since $\mathcal{K}_{\|\cdot\|} = \mathcal{K}_{M_{2d}}$ and M_{2d} is archimedean, $\mathcal{K}_{M_{2d}}$ is compact. By Jacobi's Theorem, $a + \frac{\epsilon}{2} \in M_{2d}$. So $a + \frac{\epsilon}{2} = \sum_{i=0}^{k} \sigma_i s_i$, where $\sigma_i \in \sum R^{2d}$, $i = 0, \ldots, k$, $s_0 = 1$ and $s_i \in A_{\|\cdot\|}$, $i = 1, \ldots, k$. Choose $\delta > 0$ satisfying $(\sum_{i=0}^{k} \|\sigma_i\|)\delta \leq \frac{\epsilon}{2}$. By Lemma 5.2 and continuity of the function $x \mapsto x^{2d}$ on $\tilde{R}$, there exists $r_i \in R$ such that $\|\frac{\delta}{2} + s_i - r_i^{2d}\| \leq \frac{\delta}{2}$, i.e., $\|s_i - r_i^{2d}\| \leq \delta$, $i = 1, \ldots, k$. Take $b = \sigma_0 + \sum_{i=1}^{k} \sigma_i r_i^{2d} \in \sum R^{2d}$. Then

$$\begin{array}{rcl} \|a - b\| & = & \|\sum_{i=1}^{k} \sigma_i s_i - \sum_{i=1}^{k} \sigma_i r_i^{2d} - \dfrac{\epsilon}{2}\| \\ & \leq & \sum_{i=1}^{k} \|\sigma_i\| \cdot \|s_i - r_i^{2d}\| + \dfrac{\epsilon}{2} \\ & \leq & \epsilon. \end{array}$$

This completes the proof. $\square$

Corollary 5.4. *Let $(R, \|\cdot\|)$ be a normed $\mathbb{R}$-algebra, $d \geq 1$ an integer and $L : R \longrightarrow \mathbb{R}$ a $\|\cdot\|$-continuous linear functional. If for each $a \in R$, $L(r^{2d}) \geq 0$, then there exists a Borel measure μ on $\mathcal{K}_{\|\cdot\|}$ such that*

$$L(a) = \int_{\mathcal{K}_{\|\cdot\|}} \hat{a}\ d\mu \quad \forall a \in R.$$

Proof. Since $\mathcal{K}_{\|\cdot\|}$ is compact by Theorem 5.3, the conclusion follows by applying Theorem 2.8 for $X = \mathcal{K}_{\|\cdot\|}$ and $p = 1$. $\square$

Closures of $\sum R^{2d}$-modules in $\mathcal{T}_{\mathcal{X}_R}$.

Theorem 5.5. *Let $M \subseteq R$ be a $\sum R^{2d}$-module of $\mathbb{R}$-algebra R and $d \geq 1$ an integer. Then $\overline{M}^{\mathcal{T}_{\mathcal{X}_R}} = Psd(\mathcal{K}_M)$.*

Proof. Since for each $m \in M$, $\hat{m} \geq 0$ on $\mathcal{K}_M$, we have $M \subseteq \mathrm{Psd}(\mathcal{K}_M)$. Each homomorphism $\alpha \in \mathcal{X}_R$ is continuous, so

$$\mathrm{Psd}(\mathcal{K}_M) = \bigcap_{\alpha \in \mathcal{K}_M} \alpha^{-1}([0, \infty)),$$

is closed in $\mathcal{T}_{\mathcal{X}_R}$. Therefore $\overline{M}^{\mathcal{T}_{\mathcal{X}_R}} \subseteq \mathrm{Psd}(\mathcal{K}_M)$. For the reverse inclusion, we show that for each $a \in \mathrm{Psd}(\mathcal{K}_M)$ and any open set U containing a, $U \cap M \neq \varnothing$. Hence $a \in \overline{M}^{\mathcal{T}_{\mathcal{X}_R}}$.

For $a \in \mathrm{Psd}(\mathcal{K}_M)$ and an open set U containing a, there exist $\alpha_1, \ldots, \alpha_n \in \mathcal{X}_R$ and $\epsilon > 0$ such that $U^{\epsilon}_{\alpha_1,\ldots,\alpha_n}(a) \subseteq U$, where

$$U^{\epsilon}_{\alpha_1,\ldots,\alpha_n}(a) = \{b \in R \ : \ \rho_{\alpha_i}(a - b) < \epsilon, i = 1, \ldots, n\}$$

is a typical basic open set in $\mathcal{T}_{\mathcal{X}_R}$.
Claim. For each $i = 1, \ldots, n$ there exists $t_i \in M$, $\alpha_i(a) = \alpha_i(t_i)$.

For each $i = 1, \ldots, n$, either $\alpha_i(a) \geq 0$, or $\alpha_i(a) < 0$. If $\alpha_i(a) \geq 0$, take $t_i = \alpha_i(a) \cdot 1_R$, which belonges to M. Suppose that $\alpha_i(a) < 0$. There exists $s_i \in M$ such that $\alpha_i(s_i) < 0$. Otherwise, $\alpha_i \in \mathcal{K}_M$ and hence $\alpha_i(a) \geq 0$ which is a contradiction. So $\frac{\alpha_i(a)}{\alpha_i(s_i)} > 0$ and hence $t_i = \frac{\alpha_i(a)}{\alpha_i(s_i)} s_i \in M$. This completes the proof of the Claim.

For each $1 \leq i, l \leq n$ set

$$p_{il} := \prod_{\alpha_i(t_l) \neq \alpha_j(t_l)} \frac{t_l - \alpha_j(t_l) \cdot 1}{\alpha_i(t_l) - \alpha_j(t_l)},$$

if $\alpha_i(t_l) \neq \alpha_j(t_l)$ for some $1 \leq j \leq n$, and $p_{il} = 1$, if there is no such j. Then take $p_i = \prod_{l=1}^{n} p_{il}$. Note that for each $1 \leq k \leq n$,

$$\alpha_k(p_i) = \begin{cases} 0 & \text{if } \exists l \in \{1, \ldots, n\} \quad \alpha_i(t_l) \neq \alpha_k(t_l), \\ 1 & \text{Otherwise.} \end{cases}$$

Let λ_i be the number of elements $k \in \{1,\dots,n\}$ such that $\alpha_i(t_l) = \alpha_k(t_l)$, for all $1 \le l \le n$. Take $p = \sum_{j=1}^{n} \frac{1}{\lambda_j} p_j^{2d} t_j$ which belonges to M. We have $\alpha_i(p) = \alpha_i(a)$ for $i = 1,\dots,n$ and hence $p \in U^{\epsilon}_{\alpha_1,\dots,\alpha_n}(a)$. Therefore $M \cap U^{\epsilon}_{\alpha_1,\dots,\alpha_n}(a) \neq \varnothing$, so $a \in \overline{M}^{\mathcal{T}_{\mathcal{X}_R}}$, which proves the reverse inclusion and hence $\overline{M}^{\mathcal{T}_{\mathcal{X}_R}} = \operatorname{Psd}(\mathcal{K}_M)$. □

6. Application to $R := \mathbb{R}[\underline{X}]$

We are mainly interested in the special case of real polynomials. In this case, $\mathbb{R}[\underline{X}]$ is a free finitely generated commutative $\mathbb{R}$-algebra and hence every $\alpha \in \mathcal{X}_{\mathbb{R}[\underline{X}]}$ is completely determined by $\alpha(X_i)$, $i = 1,\dots,n$. So, $\mathcal{X}_{\mathbb{R}[\underline{X}]} = \mathbb{R}^n$ with the usual euclidean topology.

Corollary 6.1. *Let K be a closed Zariski dense subset of $\mathbb{R}^n$,*

1. *The family of multiplicative seminorms $\mathcal{F}_K$ induces a lmc Hausdorff topology $\mathcal{T}_K$ on $\mathbb{R}[\underline{X}]$ such that $\overline{\sum \mathbb{R}[\underline{X}]^{2d}}^{\mathcal{T}_K} = Psd(K)$.*
2. *If K is compact subset then $\|f\|_K = \sup_{x\in K} |f(x)|$ induces a norm on $\mathbb{R}[\underline{X}]$ such that $\overline{\sum \mathbb{R}[\underline{X}]^{2d}}^{\|\cdot\|_K} = Psd(K)$.*

Proof. Apply theorems 4.5 and 4.2 to get the asserted equalities. The fact that $\mathcal{T}_K$ is Hausdorff and $\|\cdot\|_K$ is actually a norm, follows from Theorem 3.3 and Remark 3.5 respectively. □

Remark 6.2.
(i) According to [8, Theorem 3.1], the equality $\overline{\sum \mathbb{R}[\underline{X}]^{2d}}^{\|\cdot\|_K} = \operatorname{Psd}(K)$ is equivalent to the following: If L is a linear functional on $\mathbb{R}[\underline{X}]$ satisfying $L(h^{2d}) \ge 0$ for all $h \in \mathbb{R}[\underline{X}]$ and $\forall f \in \mathbb{R}[\underline{X}]$ $|L(f)| \le C\|f\|_K$ for some real $C > 0$, then there exists a positive Borel measure μ on K, representing L:

$$\forall f \in \mathbb{R}[\underline{X}] \quad L(f) = \int_K f \, d\mu.$$

(ii) Reinterpreting the equation $\overline{\sum \mathbb{R}[\underline{X}]^{2d}}^{\mathcal{T}_K} = \operatorname{Psd}(K)$, we have the following: If L be a linear functional on $\mathbb{R}[\underline{X}]$, satisfying $L(h^{2d}) \ge 0$ for all $h \in \mathbb{R}[\underline{X}]$ and for every $x \in K$, $\exists C_x > 0$ such that $|L(f)| \le C_x|f(x)|$, then there exists a positive Borel measure μ on K representing L:

$$\forall f \in \mathbb{R}[\underline{X}] \quad L(f) = \int_K f \, d\mu.$$

We now discuss the case when K is not Zariski dense. By Theorem 3.3, Φ is not injective and hence the topology $\mathcal{T}_K$ (or when K is compact the topology induced by $\|a\|_K = \sup_{\alpha\in K} |\hat{a}(\alpha)|$) will not be Hausdorff. Let $K \subseteq \mathbb{R}^n$ be given, then for $\epsilon > 0$ the set $K^{(\epsilon)} := \overline{\bigcup_{x\in K} N_\epsilon(x)}$ has nonempty

interior, in fact $K \subseteq (K^{(\epsilon)})^\circ$ and hence $K^{(\epsilon)}$ is Zariski dense in $\mathbb{R}^n$. If $0 < \epsilon_1 \le \epsilon_2$, then $K^{\epsilon_1} \subseteq K^{\epsilon 2}$ and the identity map

$$id_{\epsilon_2,\epsilon_1} : (\mathbb{R}[\underline{X}], \mathcal{T}_{K^{\epsilon_2}}) \to (\mathbb{R}[\underline{X}], \mathcal{T}_{K^{\epsilon_1}})$$

is continuous. Therefore the family $\{(\mathbb{R}[\underline{X}], \mathcal{T}_{K^{(\epsilon)}})_{\epsilon>0}, (id_{\epsilon\delta})_{\delta\le\epsilon}\}$ is an inverse system of lc and Hausdorff vector spaces. The inverse limit of this system exists and is a lc and Hausdorff space [13, Section 2.6]. Let $(V, \tau_K) = \varprojlim_{\epsilon>0}(\mathbb{R}[\underline{X}], \mathcal{T}_{K^{(\epsilon)}})$. Then $V = \mathbb{R}[\underline{X}]$ and τ_K is a lc and Hausdorff topology.

Theorem 6.3. *For any closed $K \subseteq \mathbb{R}^n$ and integer $d \ge 1$, $\overline{\sum \mathbb{R}[\underline{X}]^{2d}}^{\tau_K}$ is the cone $\tilde{Psd}(K)$, consisting of those polynomials which are non-negative over some open set, containing K.*

Proof. Since $\tau_K = \varprojlim_{\epsilon>0} \mathcal{T}_{K^{(\epsilon)}}$, we have

$$\overline{\sum \mathbb{R}[\underline{X}]^{2d}}^{\tau_K} = \varprojlim \overline{\sum \mathbb{R}[\underline{X}]^{2d}}^{\|\cdot\|_{K^{(\epsilon)}}} = \bigcap_{\epsilon} \mathrm{Psd}(K^{(\epsilon)})$$

(See [7, §4.4]) and $\bigcap_\epsilon \mathrm{Psd}(K^{(\epsilon)}) = \tilde{\mathrm{Psd}}(K)$ by definition. □

Remark 6.4. Assuming K is compact implies the compactness of each $K^{(\epsilon)}$. Therefore $\|\cdot\|_{K^{(\epsilon)}}$ is defined and and is a norm. Moreover, for $0 < \epsilon_1 \le \epsilon_2$, the identity map

$$id_{\epsilon_2,\epsilon_1} : (\mathbb{R}[\underline{X}], \|\cdot\|_{K^{(\epsilon_2)}}) \to (\mathbb{R}[\underline{X}], \|\cdot\|_{K^{(\epsilon_1)}})$$

is continuous by $\|\cdot\|_{K^{(\epsilon_1)}} \le \|\cdot\|_{K^{(\epsilon_2)}}$. So $\{(\mathbb{R}[\underline{X}], \|\cdot\|_{K^{(\epsilon)}}), (id_{\epsilon\delta})_{\delta<\epsilon}\}$ is an inverse limit of normed spaces. The inverse limit topology $\tau_K = \varprojlim_{\epsilon>0} \|\cdot\|_{K^{(\epsilon)}}$ exists and is a lc Hausdorff topology on $\mathbb{R}[\underline{X}]$. The equality $\overline{\sum \mathbb{R}[\underline{X}]^{2d}}^{\tau_K} = \tilde{\mathrm{Psd}}(K)$ can be verified similar to Theorem 6.3.

6.1. **Comparison with sub-multiplicative norm topologies.** Now, let $\|\cdot\|$ be a sub-multiplicative norm on $\mathbb{R}[\underline{X}]$, i.e.

$$\|f \cdot g\| \le \|f\| \cdot \|g\| \quad \forall f, g \in \mathbb{R}[\underline{X}].$$

Proposition 6.5. *The $\|\cdot\|$-topology is finer than $\|\cdot\|_{\mathcal{K}_{\|\cdot\|}}$-topology and*

$$\overline{\sum \mathbb{R}[\underline{X}]^{2d}}^{\|\cdot\|} = \overline{\sum \mathbb{R}[\underline{X}]^{2d}}^{\|\cdot\|_{\mathcal{K}_{\|\cdot\|}}} = Psd(\mathcal{K}_{\|\cdot\|}),$$

for every integer $d \ge 1$.

Proof. To prove that $\|\cdot\|$-topology is finer than $\|\cdot\|_{\mathcal{K}_{\|\cdot\|}}$-topology, we show $\|f\|_{\mathcal{K}_{\|\cdot\|}} \le \|f\|$. Note that $\mathcal{K}_{\|\cdot\|}$ is compact by 5.3, so $\|\cdot\|_{\mathcal{K}_{\|\cdot\|}}$ is defined and by Lemma 5.1,

$$\|f\|_{\mathcal{K}_{\|\cdot\|}} = \sup_{x\in\mathcal{K}_{\|\cdot\|}} |f(x)| \le \sup_{x\in\mathcal{K}_{\|\cdot\|}} \|f\| = \|f\|.$$

Therefore, the identity map $id : (\mathbb{R}[\underline{X}], \|\cdot\|) \to (\mathbb{R}[\underline{X}], \|\cdot\|_{\mathcal{K}_{\|\cdot\|}})$ is continuous and hence $\|\cdot\|$-topology is finer than $\|\cdot\|_{\mathcal{K}_{\|\cdot\|}}$-topology. Moreover, $\overline{\sum \mathbb{R}[\underline{X}]^{2d}}^{\|\cdot\|_{\mathcal{K}_{\|\cdot\|}}} = \mathrm{Psd}(\mathcal{K}_{\|\cdot\|})$ by Corollary 6.1. □

Definition 6.6.

(i) A function $\phi : \mathbb{N}^n \to \mathbb{R}^+$ is called an *absolute value* if

$$\phi(\underline{0}) = 1 \text{ and } \forall s,t \in \mathbb{N}^n \;\; \phi(s+t) \le \phi(s)\phi(t)\,.$$

(ii) For a polynomial $f = \sum_{s\in\mathbb{N}^n} f_s \underline{X}^s$, let $\|f\|_\phi := \sum_{s\in\mathbb{N}^n} |f_s|\phi(s)$.

If $\phi > 0$ on $\mathbb{N}^n$, then $\|\cdot\|_\phi$ defines a norm on $\mathbb{R}[\underline{X}]$. Berg and Maserick [4, 5] show that the closure of $\sum \mathbb{R}[\underline{X}]^2$ with respect to the $\|\cdot\|_\phi$-topology is $\mathrm{Psd}(\mathcal{K}_\phi)$, where $\mathcal{K}_\phi := \{x \in \mathbb{R}^n : |x^s| \le \phi(s),\ \forall s \in \mathbb{N}^n\} = \mathcal{K}_{\|\cdot\|_\phi}$, the Gelfand spectrum of $(\mathbb{R}[\underline{X}], \|\cdot\|_\phi)$.

If $\phi > 0$ then $\mathcal{K}_\phi$ has non-empty interior, and hence is Zariski dense. By [9], $\mathcal{K}_\phi$ is compact. Hence $\|\cdot\|_{\mathcal{K}_\phi}$ is defined and is a norm by Remark 3.5(1).

The following corollary to Proposition 6.5 generalizes the result of Berg and Maserick [4, Theorem 4.2.5] to the closure of $\sum \mathbb{R}[\underline{X}]^{2d}$.

Corollary 6.7. *The $\|\cdot\|_\phi$-topology is finer than $\|\cdot\|_{\mathcal{K}_\phi}$-topology and*

$$\overline{\sum \mathbb{R}[\underline{X}]^2}^{\|\cdot\|_\phi} = \overline{\sum \mathbb{R}[\underline{X}]^2}^{\|\cdot\|_{\mathcal{K}_\phi}} = Psd(\mathcal{K}_\phi).$$

6.2. **Comparison with Lasserre's topology.** Recently, Lasserre [15] considered the following $\|\cdot\|_w$ on $\mathbb{R}[\underline{X}]$:

$$\|\sum_{s\in\mathbb{N}^n} f_s \underline{X}^s\|_w = \sum_{s\in\mathbb{N}^n} |f_s| w(s),$$

where $w(s) = (2\lceil |s|/2\rceil)!$ and $|s| = |(s_1,\dots,s_n)| = s_1 + \cdots + s_n$. He proved that for any basic semi-algebraic set $K \subseteq \mathbb{R}^n$, defined by a finite set of polynomials S, the closure of the quadratic module M_S and the preordering T_S with respect to $\|\cdot\|_w$ are equal to $\mathrm{Psd}(K)$.

Proposition 6.8. *Let $K_S \subseteq \mathbb{R}^n$ be a basic closed semi-algebraic set and $d \ge 1$ an integer.*

(1) *If K_S is compact, then the $\|\cdot\|_w$-topology is finer than $\|\cdot\|_{K_S}$-topology and*

$$\overline{M_S}^{\|\cdot\|_w} = \overline{T_S}^{\|\cdot\|_w} = \overline{\sum \mathbb{R}[\underline{X}]^{2d}}^{\|\cdot\|_{K_S}} = Psd(K_S).$$

(2) *$\|\cdot\|_w$-topology is finer than $\mathcal{T}_{K_S}$ and*

$$\overline{M_S}^{\|\cdot\|_w} = \overline{T_S}^{\|\cdot\|_w} = \overline{\sum \mathbb{R}[\underline{X}]^{2d}}^{\mathcal{T}_{K_S}} = Psd(K_S).$$

Proof. (1) To show that $\|\cdot\|_w$-topology is finer than $\|\cdot\|_{K_S}$-topology, it suffices to prove that the formal identity map

$$id : (\mathbb{R}[\underline{X}], \|\cdot\|_w) \longrightarrow (\mathbb{R}[\underline{X}], \|\cdot\|_{K_S})$$

is continuous. Let $p_i : \mathbb{R}^n \to \mathbb{R}$ be the projection on i-th coordinate and

$$M = \max_{1\le i\le n}\{|p_i(x)| : x \in K_S\}.$$

So, for each $s \in \mathbb{N}^n$ we have $\|\underline{X}^s\|_{K_S} \le M^{|s|}$. Also $w(s) \ge |s|!$ for all $s \in \mathbb{N}^n$. By Stirling's formula $|s|! \sim \sqrt{2\pi} e^{(|s|+\frac{1}{2})\ln|s|-|s|}$, we see that

$$\begin{aligned} \frac{\|\underline{X}^s\|_{K_S}}{\|\underline{X}^s\|_w} &\le \frac{M^{|s|}}{|s|!} \\ &\sim \frac{1}{\sqrt{2\pi}} e^{|s|(\ln M - \ln|s|+1)-\frac{1}{2}\ln|s|} \xrightarrow{|s|\to\infty} 0. \end{aligned}$$

Therefore for some $N \in \mathbb{N}$, if $|s| > N$ then $\frac{\|\underline{X}^s\|_{K_S}}{\|\underline{X}^s\|_w} < 1$, which shows that id is bounded and hence continuous. The asserted equality follows from Corollary 4.2 and [15, Theorem 3.3].

(2) It suffices to show that for any $x \in K_S$, the evaluation map $e_x(f) = f(x)$ is $\|\cdot\|_w$-continuous. Since $\frac{|x^s|}{|s|!} \xrightarrow{|s|\to\infty} 0$, we deduce that $\sup_{f\in\mathbb{R}[\underline{X}]} \frac{|f(x)|}{\|f\|_w}$ is bounded. So, e_x and hence ρ_x is $\|\cdot\|_w$-continuous. Therefore any basic open set in $\mathcal{T}_{K_S}$ is $\|\cdot\|_w$-open. The asserted equality follows from Theorem 4.5 and [15, Theorem 3.3]. □

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[1]Department of Mathematics and Statistics,
University of Saskatchewan,
Saskatoon, SK. S7N 5E6, Canada

[2]Fachbereich Mathematik und Statistik,
Universität Konstanz
78457 Konstanz, Germany

E-mail address: `mehdi.ghasemi@usask.ca, salma.kuhlmann@uni-konstanz.de`